

# Non-Markovian Fully Coupled Forward-Backward Stochastic Systems and Classical Solutions of Path-dependent PDEs\*

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**Abstract.** This paper explores the relationship between non-Markovian fully coupled forward-backward stochastic systems and path-dependent PDEs. The definition of classical solution for the path-dependent PDE is given within the framework of functional Itô calculus. Under mild hypotheses, we prove that the forward-backward stochastic system provides the unique classical solution to the path-dependent PDE.

**Keywords:** Functional Itô calculus, Non-Markovian fully coupled forward-backward systems, Path-dependent PDEs, Classical solutions

## 1 Introduction

Linear Backward Stochastic Differential Equations (BSDEs) was introduced by Bismut [1]. The existence and uniqueness theorem for nonlinear BSDEs was established by Pardoux and Peng [13]. Then Peng [14] and Pardoux and Peng [12] gave a relationship between Markovian forward-backward systems and systems of quasilinear parabolic PDEs, which generalized the classical Feynman-Kac formula. Peng [16] pointed out that for non-Markovian forward-backward systems, it was an open problem to find the corresponding "PDE".

Recently in the framework of functional Itô calculus, a path-dependent PDE was introduced by Dupire [6] which shed light on this problem (for a recent account of this theory we refer the reader to [2], [3] and [4]). Inspired by Dupire's work, Peng and Wang [18] obtained a nonlinear Feynman-Kac formula for classical solutions of path-dependent PDEs in terms of non-Markovian BSDEs. Cosso [5] proved that, under some assumptions, the non-Markovian forward-backward system gives the unique continuous viscosity solution to the path-dependent PDE. For the further development, the readers may refer to [17] and [7].

In this paper, we study the relationship between solutions of non-Markovian fully coupled forward-backward systems and classical solutions of path-dependent PDEs. More precisely, the non-Markovian forward-backward system is described by the following fully coupled forward-backward SDE:

$$X^{\gamma_t, x}(s) = x + \int_t^s b(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s))ds + \int_t^s \sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s))dW(s), \quad (1.1)$$

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$$Y^{\gamma_t, x}(s) = g(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) - \int_s^T h(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) ds - \int_s^T Z^{\gamma_t, x}(s) dW(s), \quad s \in [t, T]. \quad (1.2)$$

We first give the definition of classical solution, within the framework of functional Itô calculus, for the path-dependent PDE. Note that in our context, we use the terminology "classical solution" to distinguish it from "viscosity solution" in [17], [7] and [5]. Then under mild hypotheses, we establish some estimates and regularity results for the solution of the above system with respect to paths. Finally, we show that the solution of (1.2) is related to the classical solution of the following path-dependent PDE

$$\begin{aligned} D_t u(\gamma_t, x) + \mathcal{L}u(\gamma_t, x) + tr[\nabla_x D_z u(\gamma_t, x) \sigma(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x))] + \frac{1}{2} tr[D_{zz} u(\gamma_t, x)] \\ = h(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x)), \\ v(\gamma_t, x) = \nabla_x u(\gamma_t, x) \sigma(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x)) + D_z u(\gamma_t, x), \\ u(\gamma_T, x) = g(\gamma_T, x), \quad \gamma_T \in \Lambda^n, \end{aligned}$$

where

$$\mathcal{L}u = \frac{1}{2} tr[(\sigma \sigma^T)(\gamma_t, x, u, v) \nabla_{xx} u] + \langle b(\gamma_t, x, u, v) \nabla_x u \rangle.$$

The paper is organized as follows: in section 2, we present some fundamental results on functional Itô calculus and FBSDE theory. We establish some estimates and regularity results for the solution of non-Markovian FBSDEs in section 3. Finally, in section 4, we give the relationship between non-Markovian fully coupled FBSDEs and path-dependent PDEs.

## 2 Preliminaries

### 2.1 Functional Itô calculus

The following notations and tools are mainly from Dupire [6]. Let  $T > 0$  be fixed. For each  $t \in [0, T]$ , we denote by  $\Lambda_t$  the set of càdlàg  $\mathbb{R}^d$ -valued functions on  $[0, t]$ . For each  $\gamma \in \Lambda_T$  the value of  $\gamma$  at time  $s \in [0, T]$  is denoted by  $\gamma(s)$ . Thus  $\gamma = \gamma(s)_{0 \leq s \leq T}$  is a càdlàg process on  $[0, T]$  and its value at time  $s$  is  $\gamma(s)$ . The path of  $\gamma$  up to time  $t$  is denoted by  $\gamma_t$ , i.e.,  $\gamma_t = \gamma(s)_{0 \leq s \leq t} \in \Lambda_t$ . We denote  $\Lambda = \bigcup_{t \in [0, T]} \Lambda_t$ . For each  $\gamma_t \in \Lambda$  and  $x \in \mathbb{R}^d$  we denote by  $\gamma_t(s)$  the value of  $\gamma_t$  at  $s \in [0, t]$  and  $\gamma_t^x := (\gamma_t(s)_{0 \leq s < t}, \gamma_t(t) + x)$  which is also an element in  $\Lambda_t$ .

Let  $|\cdot|$  denote the norm in  $\mathbb{R}^d$ . We now define a distance on  $\Lambda$ . For each  $0 \leq t, \bar{t} \leq T$  and  $\gamma_t, \bar{\gamma}_{\bar{t}} \in \Lambda$ , we denote

$$\begin{aligned} \|\gamma_t\| &:= \sup_{s \in [0, t]} |\gamma_t(s)|, \\ \|\gamma_t - \bar{\gamma}_{\bar{t}}\| &:= \sup_{s \in [0, t \vee \bar{t}]} |\gamma_t(s \wedge t) - \bar{\gamma}_{\bar{t}}(s \wedge \bar{t})|, \\ d_\infty(\gamma_t, \bar{\gamma}_{\bar{t}}) &:= \sup_{0 \leq s \leq t \vee \bar{t}} |\gamma_t(s \wedge t) - \bar{\gamma}_{\bar{t}}(s \wedge \bar{t})| + |t - \bar{t}|. \end{aligned}$$

It is obvious that  $\Lambda_t$  is a Banach space with respect to  $\|\cdot\|$  and  $d_\infty$  is not a norm.

**Definition 2.1.** A function  $u : \Lambda \mapsto \mathbb{R}$  is said to be  $\Lambda$ -continuous at  $\gamma_t \in \Lambda$ , if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that for each  $\bar{\gamma}_{\bar{t}} \in \Lambda$  with  $d_\infty(\gamma_t, \bar{\gamma}_{\bar{t}}) < \delta$ , we have  $|u(\gamma_t) - u(\bar{\gamma}_{\bar{t}})| < \varepsilon$ .  $u$  is said to be  $\Lambda$ -continuous if it is  $\Lambda$ -continuous at each  $\gamma_t \in \Lambda$ .

**Definition 2.2.** Let  $u : \Lambda \mapsto \mathbb{R}$  and  $\gamma_t \in \Lambda$  be given. If there exists  $p \in \mathbb{R}^d$ , such that

$$u(\gamma_t^x) = u(\gamma_t) + \langle p, x \rangle + o(|x|) \text{ as } x \rightarrow 0, \quad x \in \mathbb{R}^d.$$

Then we say that  $u$  is (vertically) differentiable at  $\gamma_t$  and denote the gradient of  $D_x u(\gamma_t) = p$ .  $u$  is said to be vertically differentiable in  $\Lambda$  if  $D_x u(\gamma_t)$  exists for each  $\gamma_t \in \Lambda$ . We can similarly define the Hessian  $D_{xx} u(\gamma_t)$ . It is an  $\mathbb{S}(d)$ -valued function defined on  $\Lambda$ , where  $\mathbb{S}(d)$  is the space of all  $d \times d$  symmetric matrices.

For each  $\gamma_t \in \Lambda$  we denote

$$\gamma_{t,s}(r) = \gamma_t(r) \mathbf{1}_{[0,t)}(r) + \gamma_t(t) \mathbf{1}_{[t,s]}(r), \quad r \in [0, s].$$

It is clear that  $\gamma_{t,s} \in \Lambda_s$ .

**Definition 2.3.** For a given  $\gamma_t \in \Lambda$  if we have

$$u(\gamma_{t,s}) = u(\gamma_t) + a(s - t) + o(|s - t|) \text{ as } s \rightarrow t, \quad s \geq t,$$

then we say that  $u(\gamma_t)$  is (horizontally) differentiable in  $t$  at  $\gamma_t$  and denote  $D_t u(\gamma_t) = a$ .  $u$  is said to be horizontally differentiable in  $\Lambda$  if  $D_t u(\gamma_t)$  exists for each  $\gamma_t \in \Lambda$ .

**Definition 2.4.** Define  $\mathbb{C}^{j,k}(\Lambda)$  as the set of function  $u := (u(\gamma_t))_{\gamma_t \in \Lambda}$  defined on  $\Lambda$  which are  $j$  times horizontally and  $k$  times vertically differentiable in  $\Lambda$  such that all these derivatives are  $\Lambda$ -continuous.

The following Itô formula was firstly obtained by Dupire [6] and then generalized by Cont and Fournié [2], [3] and [4].

**Theorem 2.1** (Functional Itô's formula). Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$  be a probability space, if  $X$  is a continuous semi-martingale and  $u$  is in  $\mathbb{C}^{1,2}(\Lambda)$ , then for any  $t \in [0, T]$ ,

$$\begin{aligned} u(X_t) - u(X_0) &= \int_0^t D_s u(X_s) ds + \int_0^t D_x u(X_s) dX(s) \\ &\quad + \frac{1}{2} \int_0^t D_{xx} u(X_s) d\langle X \rangle(s) \quad P - a.s. \end{aligned}$$

## 2.2 Non-Markovian fully coupled FBSDEs

Let  $\Omega = C([0, T]; \mathbb{R}^d)$  and  $P$  the Wiener measure on  $(\Omega, \mathbb{B}(\Omega))$ . We denote by  $W = (W(t))_{t \in [0, T]}$  the canonical Wiener process, with  $W(t, \omega) = \omega(t)$ ,  $t \in [0, T]$ ,  $\omega \in \Omega$ . For any  $t \in [0, T]$ , we denote by  $\mathcal{F}_t$  the  $P$ -completion of  $\sigma(W(s), s \in [0, t])$ .

For any  $t \in [0, T]$ , we denote by  $L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n)$  the set of all square integrable  $\mathcal{F}_t$ -measurable random variables,  $M^2(0, T; \mathbb{R}^n)$  the set of all  $\mathbb{R}^n$ -valued  $\mathcal{F}_t$ -adapted processes  $\vartheta(\cdot)$  such that

$$E \int_0^T |\vartheta(s)|^2 ds < +\infty.$$

Let  $t \in [0, T]$ ,  $\gamma_t \in \Lambda$  and  $x \in \mathbb{R}^n$ . We consider the following Non-Markovian forward-backward SDEs:

$$X^{\gamma_t, x}(s) = x + \int_t^s b(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr + \int_t^s \sigma(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dW(r), \quad (2.1)$$

$$Y^{\gamma_t, x}(s) = g(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) - \int_s^T h(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr - \int_s^T Z^{\gamma_t, x}(r) dW(r), \quad (2.2)$$

for every  $s \in [t, T]$ ,  $P - a.s.$ , where

$$W^{\gamma_t}(s) := \gamma(s) I_{0 \leq s < t} + (W(s) - W(t) + \gamma(t)) I_{t \leq s \leq T}.$$

We suppose that  $X(t) := xI_{0 \leq t \leq T}$ ; the processes  $X, Y, Z$  take values in  $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^{n \times d}$ ;  $b, h, \sigma$  and  $g$  take values in  $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^{n \times d}$  and  $\mathbb{R}^n$ . (2.1) and (2.2) can be rewritten as:

$$\begin{aligned} dX^{\gamma_t, x}(s) &= b(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s))ds + \sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s))dW(s), \\ dY^{\gamma_t, x}(s) &= h(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s))ds + Z^{\gamma_t, x}(s)dW(s), \\ X^{\gamma_t, x}(t) &= x, \quad Y^{\gamma_t, x}(T) = g(W_T^{\gamma_t}, X^{\gamma_t, x}(T)). \end{aligned}$$

For  $z \in \mathbb{R}^{n \times d}$ , we define  $|z| = \{tr(zz^T)\}^{1/2}$ . For  $z^1 \in \mathbb{R}^{n \times d}$ ,  $z^2 \in \mathbb{R}^{n \times d}$ ,

$$((z^1, z^2)) = tr(z^1(z^2)^T),$$

and for  $u^1 = (x^1, y^1, z^1) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,  $u^2 = (x^2, y^2, z^2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$ .

$$[u^1, u^2] = \langle x^1, x^2 \rangle + \langle y^1, y^2 \rangle + ((z^1, z^2)).$$

For  $u = (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$  and  $\gamma_t \in \Lambda^d$ , denote

$$f(\gamma_t, u) = (h(\gamma_t, u), b(\gamma_t, u), \sigma(\gamma_t, u)).$$

We give the following assumptions:

**Assumption 2.1.** For each  $u \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$ ,  $f(\cdot, u) \in M^2(0, T; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d})$ , and for each  $x \in \mathbb{R}^n$ ,  $g(\cdot, x) \in L^2(\Omega, \mathcal{F}_T; \mathbb{R}^n)$ ; there exists a constant  $c_1 > 0$ , such that

$$\begin{aligned} |f(\gamma_t, u^1) - f(\gamma_t, u^2)| &\leq c_1 |u^1 - u^2|, \quad a.e.t \in [0, T], \\ \forall \gamma_t \in \Lambda \text{ and } u^1 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}, u^2 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}; \end{aligned}$$

and

$$|g(\gamma_t, x^1) - g(\gamma_t, x^2)| \leq c_1 |x^1 - x^2|, \quad \forall \gamma_t \in \Lambda \text{ and } (x^1, x^2) \in \mathbb{R}^n \times \mathbb{R}^n.$$

**Assumption 2.2.** There exists a constant  $c_2 > 0$ , such that

$$\begin{aligned} [f(\gamma_t, u^1) - f(\gamma_t, u^2), u^1 - u^2] &\leq -c_2 |u^1 - u^2|^2, \quad a.e.t \in [0, T], \\ \forall \gamma_t \in \Lambda \text{ and } u^1 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}, u^2 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}; \end{aligned}$$

and

$$\langle g(\gamma_t, x^1) - g(\gamma_t, x^2), x^1 - x^2 \rangle \geq c_2^2 |x^1 - x^2|^2, \quad \forall \gamma_t \in \Lambda \text{ and } (x^1, x^2) \in \mathbb{R}^n \times \mathbb{R}^n.$$

**Definition 2.5.** A triple  $(X, Y, Z): [0, T] \times \Omega \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$  is called an adapted solution of the Eqs.(2.1) and (2.2), if  $(X, Y, Z) \in M^2(0, T; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d})$ , and it satisfies (2.1) and (2.2)  $P - a.s.$

Then we have the following theorem (Theorem 3.1 in [8]):

**Theorem 2.2.** Let Assumptions 2.1 and 2.2 hold, then there exists a unique adapted solution  $(X, Y, Z)$  for Eqs. (2.1) and (2.2).

For the theory of FBSDEs, the readers may refer to [11], [9], [15] and the references in [10].

### 3 Regularity

We first recall some notions in Pardoux and Peng [12].  $C^n(\mathbb{R}^p; \mathbb{R}^q)$ ,  $C_b^n(\mathbb{R}^p; \mathbb{R}^q)$ ,  $C_p^n(\mathbb{R}^p; \mathbb{R}^q)$  will denote respectively the set of functions of class  $C^n$  from  $\mathbb{R}^p$  into  $\mathbb{R}^q$ , the set of those functions of class  $C_b^n$  whose partial derivatives of order less than or equal to  $n$  are bounded, and the set of those functions of class  $C_p^n$  which, together with all their partial derivatives of order less than or equal to  $n$ , grow at most like a polynomial function of the variable  $x$  at infinity.

We give the definition of derivatives in our context.

**Definition 3.1.** An  $\mathbb{R}^n$ -valued function  $g(\gamma, x)$  on  $\Lambda_T \times \mathbb{R}^n$  is said to be in  $C^{2,2}(\Lambda_T \times \mathbb{R}^n)$ , if there exists the second order partial derivative of  $g$  in  $x$  and for each  $\gamma_t, \gamma_{\gamma_t^y} \in \Lambda_T$ ,  $t \in [0, T]$ , there exist  $p_1 \in \mathbb{R}^d$  and  $p_2 \in \mathbb{R}^d \times \mathbb{R}^d$  such that  $p_2$  is symmetric and

$$g(\gamma_{\gamma_t^y}, x) - g(\gamma_t, x) = \langle p_1, y \rangle + \frac{1}{2} \langle p_2 y, y \rangle + o(|y|^2), \quad y \in \mathbb{R}^d,$$

where  $\gamma_{\gamma_t^y} = \gamma(r)I_{[0,t)}(r) + (\gamma(r) + y)I_{[t,T]}(r)$ . We denote

$$g'_{\gamma_t}(\gamma_t, x) := p_1, \text{ and } g''_{\gamma_t}(\gamma_t, x) := p_2.$$

$g$  is said to be in  $C_{l,lip}^2(\Lambda_T)$  if  $g'_{\gamma_t}(\gamma)$  and  $g''_{\gamma_t}(\gamma)$  exist for each  $\gamma \in \Lambda_T$ , and there exists some constants  $C \geq 0$  and  $k \geq 0$  depending only on  $g$  such that for each  $\gamma, \bar{\gamma} \in \Lambda_T$ ,  $t, s \in [0, T]$ ,

$$\begin{aligned} |g(\gamma) - g(\bar{\gamma})| &\leq C(\|\gamma\|^k + \|\bar{\gamma}\|^k) \|\gamma - \bar{\gamma}\|, \\ |\Phi_{\gamma_t}(\gamma) - \Phi_{\gamma_s}(\bar{\gamma})| &\leq C(\|\gamma\|^k + \|\bar{\gamma}\|^k)(|t - s| + \|\gamma - \bar{\gamma}\|) \end{aligned}$$

with  $\Phi = g'_{\gamma_t}(\gamma), g''_{\gamma_t}(\gamma)$ . Analogously, we can define  $C^2(\Lambda_t)$ ,  $C_{l,lip}^2(\Lambda_t)$ ,  $C_{l,lip}^1(\Lambda_t)$ ,  $C_{l,lip}(\Lambda_t)$ .

Now we reconsider the solvability of equation (2.1) and (2.2). We use  $\Psi$  to represent  $b, \sigma$  or  $h$ .

**Assumption 3.1.**  $g \in C_{l,lip}^{2,2}(\Lambda_T \times \mathbb{R}^n)$  with the Lipschitz constants  $C, k$  and the first order partial derivatives in  $x$  are bounded, as well as their derivatives of order one and two with respect to  $x$ .

**Assumption 3.2.** Let  $\Psi(\gamma_t, x, y, z) = \bar{\Psi}(t, \gamma(t), x, y, z)$ , where  $\bar{\Psi} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \mapsto \mathbb{R}^n$ . Suppose that  $(t, r, x, y, z) \mapsto \bar{\Psi}(t, r, x, y, z)$  is of class  $C_p^{0,3}([0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}; \mathbb{R}^n)$  and the first order partial derivatives in  $r, x, y$  and  $z$  are bounded, as well as their derivatives of up to order two with respect to  $x, y, z$ .

By Theorem 2.1, it is easy to see that under Assumptions 2.2, 3.1 and 3.2, the FBSDE (2.1) and (2.2) has a uniqueness solution.

### 3.1 Regularity of the solution of FBSDEs

We first fix  $x$  and assume that the Lipschitz constants with respect to  $\Psi$  are  $C$  and  $k$ . Then we establish second order moment estimates for the solution of FBSDE (2.1) and (2.2).

**Lemma 3.1.** Under Assumptions 2.2, 3.1 and 3.2, there exists  $C_2$  and  $q$  depending only on  $C, T, k, x$  such that

$$\begin{aligned} E\left[\sup_{s \in [t, T]} |X^{\gamma_t, x}(s)|^2\right] &\leq C_2(1 + \|\gamma_t\|^q), \\ E\left[\sup_{s \in [t, T]} |Y^{\gamma_t, x}(s)|^2\right] &\leq C_2(1 + \|\gamma_t\|^q), \\ E\left[\left(\int_t^T |Z^{\gamma_t, x}(s)|^2 ds\right)\right] &\leq C_2(1 + \|\gamma_t\|^q). \end{aligned}$$

**Proof.** For simplicity, we only study the case  $n = d = 1$ .

Applying Itô's formula to  $(Y_{\gamma_t, x}(s))^2 e^{\beta_1 s}$  yields that

$$\begin{aligned} &(Y^{\gamma_t, x}(s))^2 e^{\beta_1 s} + \int_s^T e^{\beta_1 r} [(Z^{\gamma_t, x}(r))^2 + \beta_1 (Y^{\gamma_t, x}(r))^2] dr \\ &= g^2(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) e^{\beta_1 T} - \int_s^T 2e^{\beta_1 r} Y^{\gamma_t, x}(r) h(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr \\ &\quad - \int_s^T 2e^{\beta_1 r} Y^{\gamma_t, x}(r) Z^{\gamma_t, x}(r) dW(r). \end{aligned}$$

So

$$\begin{aligned}
& (Y^{\gamma_t, x}(s))^2 + E[\int_s^T e^{\beta_1(r-s)} [(Z^{\gamma_t, x}(r))^2 + \beta_1(Y^{\gamma_t, x}(r))^2] dr \mid \mathcal{F}_s] \\
&= E[g^2(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) e^{\beta_1(T-s)} \mid \mathcal{F}_s] \\
&\quad - E[\int_s^T 2e^{\beta_1(r-s)} Y^{\gamma_t, x}(r) h(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr \mid \mathcal{F}_s].
\end{aligned}$$

Then we have

$$\begin{aligned}
& E \sup_{t \leq s \leq T} (Y^{\gamma_t, x}(s))^2 + E[\int_t^T e^{\beta_1(r-t)} [(Z^{\gamma_t, x}(r))^2 + \beta_1(Y^{\gamma_t, x}(r))^2] dr] \\
&\leq E[g^2(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) e^{\beta_1(T-t)}] + E[\int_t^T e^{\beta_1(r-t)} \frac{2}{\beta_1} h^2(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr] \\
&\quad + E[\int_t^T e^{\beta_1(r-t)} \frac{\beta_1}{2} (Y^{\gamma_t, x}(r))^2 dr]
\end{aligned}$$

and

$$\begin{aligned}
& E \sup_{t \leq s \leq T} (Y^{\gamma_t, x}(s))^2 + E[\int_t^T e^{\beta_1(r-t)} [(Z^{\gamma_t, x}(r))^2 + \frac{\beta_1}{2} (Y^{\gamma_t, x}(r))^2] dr] \\
&\leq E[g^2(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) e^{\beta_1(T-t)}] + E[\int_t^T e^{\beta_1(r-t)} \frac{2}{\beta_1} h^2(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr].
\end{aligned} \tag{3.1}$$

Applying Itô's formula to  $(X^{\gamma_t, x}(s))^2$  yields that

$$\begin{aligned}
& (X^{\gamma_t, x}(s))^2 \\
&= x^2 + \int_t^s 2X^{\gamma_t, x}(r) b(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr + \int_t^s \sigma^2(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr \\
&\quad + \int_t^s 2X^{\gamma_t, x}(r) \sigma(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dW(r).
\end{aligned}$$

By inequality  $2ab \leq a^2 + b^2$  and Burkholder-Davis-Gundy's inequality, there is a constant  $C_0$  such that

$$\begin{aligned}
& E \sup_{t \leq s \leq T} (X^{\gamma_t, x}(s))^2 \\
&\leq C_0 [x^2 + E \int_t^T (X^{\gamma_t, x}(s))^2 ds + E \int_t^T b^2(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) ds \\
&\quad + E \int_t^T \sigma^2(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) ds].
\end{aligned} \tag{3.2}$$

Applying Itô formula to  $X^{\gamma_t, x}(s)Y^{\gamma_t, x}(s)$ ,

$$\begin{aligned}
& X^{\gamma_t, x}(T)Y^{\gamma_t, x}(T) - X^{\gamma_t, x}(t)Y^{\gamma_t, x}(t) \\
&= \int_t^T X^{\gamma_t, x}(r) h(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr \\
&\quad + \int_t^T X^{\gamma_t, x}(r) Z^{\gamma_t, x}(r) dW(r) \\
&\quad + \int_t^T Y^{\gamma_t, x}(r) b(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr \\
&\quad + \int_t^T Z^{\gamma_t, x}(r) \sigma(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr \\
&\quad + \int_t^T Y^{\gamma_t, x}(r) \sigma(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dW(r).
\end{aligned}$$

Set

$$0^{\gamma_t, x}(s) = \gamma_t(s) I_{0 \leq s \leq t} + 0 I_{t < s \leq T}(s).$$

We have

$$\begin{aligned}
& (X^{\gamma_t, x}(T) - 0^{\gamma_t, x}(T))(g(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) - g(W_T^{\gamma_t}, 0^{\gamma_t, x}(T))) \\
&+ (X^{\gamma_t, x}(T) - 0^{\gamma_t, x}(T))g(W_T^{\gamma_t}, 0) - X^{\gamma_t, x}(t)Y^{\gamma_t, x}(t) \\
&= \int_t^T [(X^{\gamma_t, x}(r) - 0^{\gamma_t, x}(r))(h(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) - h(W_r^{\gamma_t}, 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r))) \\
&\quad + (X^{\gamma_t, x}(r) - 0^{\gamma_t, x}(r))h(W_r^{\gamma_t}, 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r))] dr \\
&\quad + \int_t^T Y^{\gamma_t, x}(r) \sigma(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dW(r) + \int_t^T X^{\gamma_t, x}(r) Z^{\gamma_t, x}(r) dW(r) \\
&\quad + \int_t^T [(Y^{\gamma_t, x}(r) - 0^{\gamma_t, x}(r))(b(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) - b(W_r^{\gamma_t}, 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r))) \\
&\quad + (Y^{\gamma_t, x}(r) - 0^{\gamma_t, x}(r))b(W_r^{\gamma_t}, 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r))] dr \\
&\quad + \int_t^T [(Z^{\gamma_t, x}(r) - 0^{\gamma_t, x}(r))(\sigma(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) - \sigma(W_r^{\gamma_t}, 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r))) \\
&\quad + (Z^{\gamma_t, x}(r) - 0^{\gamma_t, x}(r))\sigma(W_r^{\gamma_t}, 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r), 0^{\gamma_t, x}(r))] dr.
\end{aligned}$$

Then by the Assumption 2.1, for enough small  $\varepsilon > 0$ ,

$$\begin{aligned}
& (C_2 - \varepsilon) E\{(X^{\gamma_t, x}(T))^2 + \int_t^T [(X^{\gamma_t, x}(r))^2 + (Y^{\gamma_t, x}(r))^2 + (Z^{\gamma_t, x}(r))^2] dr\} \\
&\leq C_\varepsilon + \frac{x}{\beta_3} + \beta_3 Y^{\gamma_t, x}(t).
\end{aligned} \tag{3.3}$$

Taking  $\beta_1 = 4C^2 + 1$ , and  $q = 2(1 + k)$ , from (3.1), (3.3) and (3.2), (3.3), we derive (note that  $C_\varepsilon$  will change line by line)

$$\begin{aligned} & E \sup_{t \leq s \leq T} (Y^{\gamma_t, x}(s))^2 + E \left[ \int_t^T [(Z^{\gamma_t, x}(r))^2 + (Y^{\gamma_t, x}(r))^2] dr \right] \\ & \leq C_\varepsilon (1 + \|\gamma_t\|^q + \frac{x}{\beta_3} + \beta_3 Y^{\gamma_t, x}(t)). \end{aligned} \quad (3.4)$$

We also have

$$E \sup_{t \leq s \leq T} (X^{\gamma_t, x}(s))^2 \leq C_\varepsilon (1 + \frac{x}{\beta_3} + \|\gamma_t\|^q + \beta_3 Y^{\gamma_t, x}(t)). \quad (3.5)$$

By (3.4) and (3.5), we have

$$\begin{aligned} & E [\sup_{t \leq s \leq T} (Y^{\gamma_t, x}(s))^2 + \sup_{t \leq s \leq T} (X^{\gamma_t, x}(s))^2] + E \left[ \int_t^T [(Z^{\gamma_t, x}(r))^2 + (Y^{\gamma_t, x}(r))^2] dr \right] \\ & \leq C_\varepsilon (1 + \|\gamma_t\|^q + \frac{x}{\beta_3} + \beta_3 Y^{\gamma_t, x}(t)). \end{aligned}$$

Finally taking  $\beta_3$  enough small, we get the result.  $\square$

Now we study the regularity properties of the solution of FBSDE (2.1) and (2.2) with respect to the "parameter"  $\gamma_t$ . For  $0 \leq s < t \leq T$ , set  $Y^{\gamma_t, x}(s) = Y^{\gamma_t, x}(s \vee t)$  and  $Z^{\gamma_t, x}(s) = 0$ .

**Theorem 3.1.** *Under Assumptions 2.2, 3.2, 3.3, there exists  $C_2$  and  $q$  depending only on  $C, c_2, x$  such that for any  $t, \bar{t} \in [0, T]$ ,  $\gamma_t, \bar{\gamma}_{\bar{t}}$ , and  $h, \bar{h} \in \mathbb{R} \setminus \{0\}$ .*

(i)

$$E \left[ \sup_{u \in [t \vee \bar{t}, T]} |Y^{\gamma_t, x}(u) - Y^{\bar{\gamma}_{\bar{t}}, x}(u)|^2 \right] \leq C_2 (1 + \|\gamma_t\|^q + \|\bar{\gamma}_{\bar{t}}\|^q) (\|\gamma_t - \bar{\gamma}_{\bar{t}}\|^2 + |t - \bar{t}|),$$

(ii)

$$E \left[ \sup_{u \in [t \vee \bar{t}, T]} |X^{\gamma_t, x}(u) - X^{\bar{\gamma}_{\bar{t}}, x}(u)|^2 \right] \leq C_2 (1 + \|\gamma_t\|^q + \|\bar{\gamma}_{\bar{t}}\|^q) (\|\gamma_t - \bar{\gamma}_{\bar{t}}\|^2 + |t - \bar{t}|),$$

(iii)

$$E \left[ \int_{t \vee \bar{t}}^T |Z^{\gamma_t, x}(u) - Z^{\bar{\gamma}_{\bar{t}}, x}(u)|^2 du \right] \leq C_2 (1 + \|\gamma_t\|^q + \|\bar{\gamma}_{\bar{t}}\|^q) (\|\gamma_t - \bar{\gamma}_{\bar{t}}\|^2 + |t - \bar{t}|),$$

(iv)

$$\begin{aligned} & E [\sup_{u \in [t \vee \bar{t}, T]} |\Delta_h^i Y^{\gamma_t, x}(u) - \Delta_h^i Y^{\bar{\gamma}_{\bar{t}}, x}(u)|^2] \\ & \leq C_2 (1 + \|\gamma_t\|^q + \|\bar{\gamma}_{\bar{t}}\|^q + |h|^q + |\bar{h}|^q) (|h - \bar{h}|^2 + \|\gamma_t - \bar{\gamma}_{\bar{t}}\|^2 + |t - \bar{t}|), \end{aligned}$$

(v)

$$\begin{aligned} & E [\sup_{u \in [t \vee \bar{t}, T]} |\Delta_h^i X^{\gamma_t, x}(u) - \Delta_h^i X^{\bar{\gamma}_{\bar{t}}, x}(u)|^2] \\ & \leq C_2 (1 + \|\gamma_t\|^q + \|\bar{\gamma}_{\bar{t}}\|^q + |h|^q + |\bar{h}|^q) (|h - \bar{h}|^2 + \|\gamma_t - \bar{\gamma}_{\bar{t}}\|^2 + |t - \bar{t}|), \end{aligned}$$

(vi)

$$\begin{aligned} & E [\int_{t \vee \bar{t}}^T |\Delta_h^i Z^{\gamma_t, x}(u) - \Delta_h^i Z^{\bar{\gamma}_{\bar{t}}, x}(u)|^2 du] \\ & \leq C_2 (1 + \|\gamma_t\|^q + \|\bar{\gamma}_{\bar{t}}\|^q + |h|^q + |\bar{h}|^q) (|h - \bar{h}|^2 + \|\gamma_t - \bar{\gamma}_{\bar{t}}\|^2 + |t - \bar{t}|), \end{aligned}$$

where

$$\begin{aligned} \Delta_h^i X^{\gamma_t, x}(s) &= \frac{1}{h} (X^{\gamma_t^{h e_i}, x}(s) - X^{\gamma_t, x}(s)), \\ \Delta_h^i Y^{\gamma_t, x}(s) &= \frac{1}{h} (Y^{\gamma_t^{h e_i}, x}(s) - Y^{\gamma_t, x}(s)), \Delta_h^i Z^{\gamma_t, x}(s) = \frac{1}{h} (Z^{\gamma_t^{h e_i}, x}(s) - Z^{\gamma_t, x}(s)) \end{aligned}$$

and  $(e_1, \dots, e_d)$  is an orthonormal basis of  $\mathbb{R}^d$ .

**Proof.**  $(X^{\gamma_t, x} - X^{\bar{\gamma}_t, x}, Y^{\gamma_t, x} - Y^{\bar{\gamma}_t, x}, Z^{\gamma_t, x} - Z^{\bar{\gamma}_t, x})$  can be formed as a linearized FBSDE: for each  $s \in [t, T]$  and  $\bar{t} \leq t$ ,

$$\begin{aligned}
& X^{\gamma_t, x}(s) - X^{\bar{\gamma}_t, x}(s) \\
&= \int_{\bar{t}}^t b(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr \\
&\quad + \int_t^s b(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) - b(W_r^{\bar{\gamma}_t}, X^{\bar{\gamma}_t, x}(r), Y^{\bar{\gamma}_t, x}(r), Z^{\bar{\gamma}_t, x}(r)) dr \\
&\quad + \int_t^s \sigma(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) - \sigma(W_r^{\bar{\gamma}_t}, X^{\bar{\gamma}_t, x}(r), Y^{\bar{\gamma}_t, x}(r), Z^{\bar{\gamma}_t, x}(r)) dW(r) \\
&= \int_{\bar{t}}^t b(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) dr + \int_t^s [\alpha_{\gamma_t, \bar{\gamma}_t}(r) + \theta_{\gamma_t, \bar{\gamma}_t}(r)(X^{\gamma_t, x}(r) - X^{\bar{\gamma}_t, x}(r)) \\
&\quad + \beta_{\gamma_t, \bar{\gamma}_t}(r)(Y^{\gamma_t, x}(r) - Y^{\bar{\gamma}_t, x}(r)) + \delta_{\gamma_t, \bar{\gamma}_t}(r)(Z^{\gamma_t, x}(r) - Z^{\bar{\gamma}_t, x}(r))] dr \\
&\quad + \int_t^s [\bar{\alpha}_{\gamma_t, \bar{\gamma}_t}(r) + \bar{\theta}_{\gamma_t, \bar{\gamma}_t}(r)(X^{\gamma_t, x}(r) - X^{\bar{\gamma}_t, x}(r)) + \bar{\beta}_{\gamma_t, \bar{\gamma}_t}(r)(Y^{\gamma_t, x}(r) \\
&\quad - Y^{\bar{\gamma}_t, x}(r)) + \bar{\delta}_{\gamma_t, \bar{\gamma}_t}(r)(Z^{\gamma_t, x}(r) - Z^{\bar{\gamma}_t, x}(r))] dW(r),
\end{aligned}$$

and

$$\begin{aligned}
& Y^{\gamma_t, x}(s) - Y^{\bar{\gamma}_t, x}(s) \\
&= g(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) - g(W_T^{\bar{\gamma}_t}, X^{\bar{\gamma}_t, x}(T)) + \int_s^T [h(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) \\
&\quad - h(W_r^{\bar{\gamma}_t}, X^{\bar{\gamma}_t, x}(r), Y^{\bar{\gamma}_t, x}(r), Z^{\bar{\gamma}_t, x}(r))] dr + \int_s^T (Z^{\gamma_t, x}(r) - Z^{\bar{\gamma}_t, x}(r)) dW(r) \\
&= g(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) - g(W_T^{\bar{\gamma}_t}, X^{\bar{\gamma}_t, x}(T)) - \int_s^T [\hat{\alpha}_{\gamma_t, \bar{\gamma}_t}(r) + \hat{\theta}_{\gamma_t, \bar{\gamma}_t}(r)(X^{\gamma_t, x}(r) - X^{\bar{\gamma}_t, x}(r)) \\
&\quad + \hat{\beta}_{\gamma_t, \bar{\gamma}_t}(r)(Y^{\gamma_t, x}(r) - Y^{\bar{\gamma}_t, x}(r)) + \hat{\delta}_{\gamma_t, \bar{\gamma}_t}(r)(Z^{\gamma_t, x}(r) - Z^{\bar{\gamma}_t, x}(r))] dr + \int_s^T (Z^{\gamma_t, x}(r) - Z^{\bar{\gamma}_t, x}(r)) dW(r).
\end{aligned}$$

Here

$$\begin{aligned}
\alpha_{\gamma_t, \bar{\gamma}_t}(r) &= b(W_r^{\gamma_t}, X^{\bar{\gamma}_t, x}(r), Y^{\bar{\gamma}_t, x}(r), Z^{\bar{\gamma}_t, x}(r)) - b(W_r^{\bar{\gamma}_t}, X^{\bar{\gamma}_t, x}(r), Y^{\bar{\gamma}_t, x}(r), Z^{\bar{\gamma}_t, x}(r)), \\
\theta_{\gamma_t, \bar{\gamma}_t}(r) &= \int_0^1 \frac{\partial b}{\partial x}(W_r^{\gamma_t}, U^{\bar{\gamma}_t, x}(r) + \theta(U^{\gamma_t}(r) - U^{\bar{\gamma}_t, x}(r))) d\theta, \\
\beta_{\gamma_t, \bar{\gamma}_t}(r) &= \int_0^1 \frac{\partial b}{\partial y}(W_r^{\gamma_t}, U^{\bar{\gamma}_t, x}(r) + \theta(U^{\gamma_t}(r) - U^{\bar{\gamma}_t, x}(r))) d\theta, \\
\delta_{\gamma_t, \bar{\gamma}_t}(r) &= \int_0^1 \frac{\partial b}{\partial z}(W_r^{\gamma_t}, U^{\bar{\gamma}_t, x}(r) + \theta(U^{\gamma_t}(r) - U^{\bar{\gamma}_t, x}(r))) d\theta, \\
\bar{\alpha}_{\gamma_t, \bar{\gamma}_t}(r) &= \sigma(W_r^{\gamma_t}, X^{\bar{\gamma}_t, x}(r), Y^{\bar{\gamma}_t, x}(r), Z^{\bar{\gamma}_t, x}(r)) - \sigma(W_r^{\bar{\gamma}_t}, X^{\bar{\gamma}_t, x}(r), Y^{\bar{\gamma}_t, x}(r), Z^{\bar{\gamma}_t, x}(r)), \\
\bar{\theta}_{\gamma_t, \bar{\gamma}_t}(r) &= \int_0^1 \frac{\partial \sigma}{\partial x}(W_r^{\gamma_t}, U^{\bar{\gamma}_t, x}(r) + \theta(U^{\gamma_t}(r) - U^{\bar{\gamma}_t, x}(r))) d\theta, \\
\bar{\beta}_{\gamma_t, \bar{\gamma}_t}(r) &= \int_0^1 \frac{\partial \sigma}{\partial y}(W_r^{\gamma_t}, U^{\bar{\gamma}_t, x}(r) + \theta(U^{\gamma_t}(r) - U^{\bar{\gamma}_t, x}(r))) d\theta, \\
\bar{\delta}_{\gamma_t, \bar{\gamma}_t}(r) &= \int_0^1 \frac{\partial \sigma}{\partial z}(W_r^{\gamma_t}, U^{\bar{\gamma}_t, x}(r) + \theta(U^{\gamma_t}(r) - U^{\bar{\gamma}_t, x}(r))) d\theta, \\
\hat{\alpha}_{\gamma_t, \bar{\gamma}_t}(r) &= h(W_r^{\gamma_t}, X^{\bar{\gamma}_t, x}(r), Y^{\bar{\gamma}_t, x}(r), Z^{\bar{\gamma}_t, x}(r)) - h(W_r^{\bar{\gamma}_t}, X^{\bar{\gamma}_t, x}(r), Y^{\bar{\gamma}_t, x}(r), Z^{\bar{\gamma}_t, x}(r)), \\
\hat{\theta}_{\gamma_t, \bar{\gamma}_t}(r) &= \int_0^1 \frac{\partial h}{\partial x}(W_r^{\gamma_t}, U^{\bar{\gamma}_t, x}(r) + \theta(U^{\gamma_t}(r) - U^{\bar{\gamma}_t, x}(r))) d\theta, \\
\hat{\beta}_{\gamma_t, \bar{\gamma}_t}(r) &= \int_0^1 \frac{\partial h}{\partial y}(W_r^{\gamma_t}, U^{\bar{\gamma}_t, x}(r) + \theta(U^{\gamma_t}(r) - U^{\bar{\gamma}_t, x}(r))) d\theta, \\
\hat{\delta}_{\gamma_t, \bar{\gamma}_t}(r) &= \int_0^1 \frac{\partial h}{\partial z}(W_r^{\gamma_t}, U^{\bar{\gamma}_t, x}(r) + \theta(U^{\gamma_t}(r) - U^{\bar{\gamma}_t, x}(r))) d\theta,
\end{aligned}$$

where  $U^{\gamma_t, x} = (X^{\gamma_t, x}, Y^{\gamma_t, x}, Z^{\gamma_t, x})$ . Under Assumptions 2.2, 3.2 and 3.3, using the similar methods as in Lemma 3.1, we get the first three inequalities. For the next three inequalities, we can write  $(\Delta_h^i X^{\gamma_t, x}, \Delta_h^i Y^{\gamma_t, x}, \Delta_h^i Z^{\gamma_t, x})$  as the solution of the following linearized FBSDE:

$$\begin{aligned}
\Delta_h^i X^{\gamma_t, x}(s) &= \int_t^s \left[ \frac{1}{h} \alpha_{\gamma_t, \gamma_t^{h_{e_i}}}(r) + \theta_{\gamma_t, \gamma_t^{h_{e_i}}}(r) \Delta_h^i X^{\gamma_t, x}(r) + \beta_{\gamma_t, \gamma_t^{h_{e_i}}}(r) \Delta_h^i Y^{\gamma_t, x}(r) + \delta_{\gamma_t, \gamma_t^{h_{e_i}}}(r) \Delta_h^i Z^{\gamma_t, x}(r) \right] dr \\
&\quad + \int_t^s \left[ \frac{1}{h} \bar{\alpha}_{\gamma_t, \gamma_t^{h_{e_i}}}(r) + \bar{\theta}_{\gamma_t, \gamma_t^{h_{e_i}}}(r) \Delta_h^i X^{\gamma_t, x}(r) + \bar{\beta}_{\gamma_t, \gamma_t^{h_{e_i}}}(r) \Delta_h^i Y^{\gamma_t, x}(r) + \bar{\delta}_{\gamma_t, \gamma_t^{h_{e_i}}}(r) \Delta_h^i Z^{\gamma_t, x}(r) \right] dW(r), \\
\Delta_h^i Y^{\gamma_t, x}(s) &= \frac{1}{h} (g(W_T^{\gamma_t^{h_{e_i}}}, X^{\gamma_t^{h_{e_i}}, x}(T)) - g(W_T^{\gamma_t}, X^{\gamma_t, x}(T))) - \int_s^T \left[ \frac{1}{h} \hat{\alpha}_{\gamma_t, \gamma_t^{h_{e_i}}}(r) + \hat{\theta}_{\gamma_t, \gamma_t^{h_{e_i}}}(r) \Delta_h^i X^{\gamma_t, x}(r) \right. \\
&\quad \left. + \hat{\beta}_{\gamma_t, \gamma_t^{h_{e_i}}}(r) \Delta_h^i Y^{\gamma_t, x}(r) + \hat{\delta}_{\gamma_t, \gamma_t^{h_{e_i}}}(r) \Delta_h^i Z^{\gamma_t, x}(r) \right] dr - \int_s^T \Delta_h^i Z^{\gamma_t, x}(r) dW(r),
\end{aligned}$$

Then the same calculus yields that

$$E \left[ \sup_{s \in [t, T]} |\Delta_h^i Y^{\gamma_t, x}(s)|^2 + \sup_{s \in [t, T]} |\Delta_h^i X^{\gamma_t, x}(s)|^2 + \left| \int_t^T |\Delta_h^i Z^{\gamma_t, x}(r)|^2 dr \right| \right] \leq C_2 (1 + \|\gamma_t\|^q + |h|^q).$$



Notice that

$$\begin{aligned}
& \Delta_h^i X^{\gamma_t, x}(s) - \Delta_h^i X^{\bar{\gamma}_t, x}(s) \\
&= \int_t^s [\frac{1}{h} \alpha_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \frac{1}{h} \alpha_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) + \theta_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \Delta_h^i X^{\gamma_t, x}(r) - \theta_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i X^{\bar{\gamma}_t, x}(r) \\
&+ \beta_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \Delta_h^i Y^{\gamma_t, x}(r) - \beta_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i Y^{\bar{\gamma}_t, x}(r) + \delta_{\gamma_t, \gamma_t}^{h_{e_i}} \Delta_h^i Z^{\gamma_t, x}(r) - \delta_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i Z^{\bar{\gamma}_t, x}(r)] dr \\
&+ \int_t^s [\frac{1}{h} \bar{\alpha}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \frac{1}{h} \bar{\alpha}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) + \bar{\theta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \Delta_h^i X^{\gamma_t, x}(r) - \bar{\theta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i X^{\bar{\gamma}_t, x}(r) \\
&+ \bar{\beta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \Delta_h^i Y^{\gamma_t, x}(r) - \bar{\beta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i Y^{\bar{\gamma}_t, x}(r) + \bar{\delta}_{\gamma_t, \gamma_t}^{h_{e_i}} \Delta_h^i Z^{\gamma_t, x}(r) - \bar{\delta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i Z^{\bar{\gamma}_t, x}(r)] dW(r),
\end{aligned}$$

and

$$\begin{aligned}
& \Delta_h^i Y^{\gamma_t, x}(s) - \Delta_h^i Y^{\bar{\gamma}_t, x}(s) \\
&= \frac{1}{h} (g(W_T^{\gamma_t^{h_{e_i}}}, X^{\gamma_t^{h_{e_i}}, x}(T)) - g(W_T^{\bar{\gamma}_t^{\bar{h}_{e_i}}}, X^{\bar{\gamma}_t^{\bar{h}_{e_i}}, x}(T))) - \frac{1}{h} (g(X_T^{\gamma_t^{\bar{h}_{e_i}}}, X^{\gamma_t^{\bar{h}_{e_i}}, x}(T)) - g(W_T^{\bar{\gamma}_t}, X^{\bar{\gamma}_t, x}(T))) \\
&- \{ \int_s^T [\frac{1}{h} \hat{\alpha}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \frac{1}{h} \hat{\alpha}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) + \hat{\theta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \Delta_h^i X^{\gamma_t, x}(r) - \hat{\theta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i X^{\bar{\gamma}_t, x}(r) \\
&+ \hat{\beta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \Delta_h^i Y^{\gamma_t, x}(r) - \hat{\beta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i Y^{\bar{\gamma}_t, x}(r) + \hat{\delta}_{\gamma_t, \gamma_t}^{h_{e_i}} \Delta_h^i Z^{\gamma_t, x}(r) - \hat{\delta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i Z^{\bar{\gamma}_t, x}(r)] dr \} \\
&- \int_s^T (\Delta_h^i Z^{\gamma_t, x}(r) - \Delta_h^i Z^{\bar{\gamma}_t, x}(r)) dW(r).
\end{aligned}$$

Then

$$(\tilde{X}(s), \tilde{Y}(s), \tilde{Z}(s)) := (\Delta_h^i X^{\gamma_t, x}(s) - \Delta_h^i X^{\bar{\gamma}_t, x}(s), \Delta_h^i Y^{\gamma_t, x}(s) - \Delta_h^i Y^{\bar{\gamma}_t, x}(s), \Delta_h^i Z^{\gamma_t, x}(s) - \Delta_h^i Z^{\bar{\gamma}_t, x}(s))$$

solves the FBSDE

$$\begin{aligned}
\tilde{X}(s) &= \int_t^s [\theta_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \tilde{X}(r) + \beta_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \tilde{Y}(r) + \delta_{\gamma_t, \gamma_t}^{h_{e_i}} \tilde{Z}(r) + \tilde{b}(r)] dr \\
&+ \int_t^s [\bar{\theta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \tilde{Y}(r) + \bar{\beta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \tilde{Y}(r) + \bar{\delta}_{\gamma_t, \gamma_t}^{h_{e_i}} \tilde{Z}(r) + \tilde{\sigma}(r)] dW(r),
\end{aligned}$$

$$\begin{aligned}
\tilde{Y}(s) &= \frac{1}{h} (g(W_T^{\gamma_t^{h_{e_i}}}, X^{\gamma_t^{h_{e_i}}, x}(T)) - g(W_T^{\bar{\gamma}_t^{\bar{h}_{e_i}}}, X^{\bar{\gamma}_t^{\bar{h}_{e_i}}, x}(T))) - \frac{1}{h} (g(W_T^{\gamma_t^{\bar{h}_{e_i}}}, X^{\gamma_t^{\bar{h}_{e_i}}, x}(T)) - g(W_T^{\bar{\gamma}_t}, X^{\bar{\gamma}_t, x}(T))) \\
&- \int_s^T [\hat{\theta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \tilde{X}(r) + \hat{\beta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) \tilde{Y}(r) + \hat{\delta}_{\gamma_t, \gamma_t}^{h_{e_i}} \tilde{Z}(r) + \tilde{h}(r)] dr - \int_s^T \tilde{Z}(r) dW(r),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{b}(r) &= [\beta_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \beta_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r)] \Delta_h^i Y^{\gamma_t, x}(r) + \theta_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \theta_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i X^{\gamma_t, x}(r) \\
&+ [\delta_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \delta_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r)] \Delta_h^i Z^{\gamma_t, x}(r) + \frac{1}{h} \alpha_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \frac{1}{h} \alpha_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r), \\
\tilde{\sigma}(r) &= [\bar{\beta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \bar{\beta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r)] \Delta_h^i Y^{\bar{\gamma}_t, x}(r) + \bar{\beta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \bar{\beta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i X^{\bar{\gamma}_t, x}(r) \\
&+ [\bar{\delta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \bar{\delta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r)] \Delta_h^i Z^{\bar{\gamma}_t, x}(r) + \frac{1}{h} \bar{\alpha}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \frac{1}{h} \bar{\alpha}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r), \\
\tilde{h}(r) &= [\hat{\beta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \hat{\beta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r)] \Delta_h^i Y^{\gamma_t, x}(r) + \hat{\beta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \hat{\beta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r) \Delta_h^i X^{\gamma_t, x}(r) \\
&+ [\hat{\delta}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \hat{\delta}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r)] \Delta_h^i Z^{\gamma_t, x}(r) + \frac{1}{h} \hat{\alpha}_{\gamma_t, \gamma_t}^{h_{e_i}}(r) - \frac{1}{h} \hat{\alpha}_{\bar{\gamma}_t, \bar{\gamma}_t}^{\bar{h}_{e_i}}(r).
\end{aligned}$$

Thus, under Assumption 2.2, 3.2 and 3.3, similarly as in Lemma 3.1, we can get the last three inequalities.

□

**Theorem 3.2.** For each  $\gamma_t \in \Lambda$ ,  $\{Y^{\gamma_t, z}(s), s \in [0, T], z \in \mathbb{R}^d\}$  has a version which is a.e. of class  $C^{0,2}([0, T] \times \mathbb{R}^d)$ .

**Proof.** In order to simplify the presentation, we only consider  $n = d = 1$ . Applying Lemma 3.1, then for each  $h, \bar{h} \in \mathbb{R} \setminus \{0\}$  and  $k, \bar{k} \in \mathbb{R}$ ,

$$\begin{aligned}
& E[\sup_{u \in [t, T]} |Y^{\gamma_t^k, x}(u) - Y^{\gamma_t^{\bar{k}}, x}(u)|^2] \leq C_2(1 + \|\gamma_t\|^q) |k - \bar{k}|^2, \\
& E[\sup_{u \in [t, T]} |X^{\gamma_t^k, x}(u) - X^{\gamma_t^{\bar{k}}, x}(u)|^2] \leq C_2(1 + \|\gamma_t\|^q) |k - \bar{k}|^2, \\
& E[\int_t^T |Z^{\gamma_t^k, x}(u) - Z^{\gamma_t^{\bar{k}}, x}(u)|^2 du] \leq C_2(1 + \|\gamma_t\|^q) |k - \bar{k}|^2, \\
& E[\sup_{u \in [t, T]} |\Delta_h^i Y^{\gamma_t^k, x}(u) - \Delta_{\bar{h}}^i Y^{\gamma_t^{\bar{k}}, x}(u)|^2] \\
& \leq C_2(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q + |h|^q + |\bar{h}|^q)(|k - \bar{k}|^2 + |h - \bar{h}|^2), \\
& E[\sup_{u \in [t, T]} |\Delta_h^i X^{\gamma_t^k, x}(u) - \Delta_{\bar{h}}^i X^{\gamma_t^{\bar{k}}, x}(u)|^2] \\
& \leq C_2(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q + |h|^q + |\bar{h}|^q)(|k - \bar{k}|^2 + |h - \bar{h}|^2), \\
& E[\int_t^T |\Delta_h^i Z^{\gamma_t^k, x}(u) - \Delta_{\bar{h}}^i Z^{\gamma_t^{\bar{k}}, x}(u)|^2 du] \\
& \leq C_2(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q + |h|^q + |\bar{h}|^q)(|k - \bar{k}|^2 + |h - \bar{h}|^2).
\end{aligned}$$

By kolmogorov's criterion, there exists a continuous derivative of  $Y^{\gamma_t^z, x}(s)$  ( $X^{\gamma_t^z, x}(s)$ ) with respect to  $z$ . There also exists a mean-square derivative of  $Z^{\gamma_t^z, x}(s)$  with respect to  $z$ , which is mean square continuous in  $z$ . We denote them by

$$(D_z Y^{\gamma_t, x}, D_z X^{\gamma_t, x}, D_z Z^{\gamma_t, x}).$$

By Theorem 3.1 and Definition 3.1,  $(D_z Y^{\gamma_t, x}, D_z X^{\gamma_t, x}, D_z Z^{\gamma_t, x})$  is the solution of the following FBSDE:

$$\begin{aligned}
& D_z X^{\gamma_t, x}(s) \\
& = \int_t^s [b'_{\gamma_t}(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) + b'_x(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) D_z X^{\gamma_t, x}(r) \\
& \quad + b'_y(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) D_z Y^{\gamma_t, x}(r) + b'_z(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) D_z Z^{\gamma_t, x}(r)] dr \\
& \quad + \int_t^s [\sigma'_{\gamma_t}(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) + \sigma'_x(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) D_z X^{\gamma_t, x}(r) \\
& \quad + \sigma'_y(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) D_z Y^{\gamma_t, x}(r) + \sigma'_z(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) D_z Z^{\gamma_t, x}(r)] dW(r), \\
& D_z Y^{\gamma_t, x}(s) \\
& = g'_{\gamma_t}(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) + g'_x(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) D_z X^{\gamma_t, x}(T) - \int_s^T [h'_{\gamma_t}(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) \\
& \quad + h'_x(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) D_z X^{\gamma_t, x}(r) + h'_y(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) D_z Y^{\gamma_t, x}(r) \\
& \quad + h'_z(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) D_z Z^{\gamma_t, x}(r)] dr - \int_s^T D_z Z^{\gamma_t, x}(r) dW(r).
\end{aligned}$$

It is easy to check that the above FBSDE satisfies Assumptions 2.2, 3.2 and 3.3. Then the above FBSDE has a uniqueness solution. Thus, the existence of a continuous second order derivative of  $Y^{\gamma_t^z, x}(s)$  with respect to  $z$  can be proved in a similar way.  $\square$

Define

$$u(\gamma_t, x) := Y^{\gamma_t, x}(t), \quad \text{for } \gamma_t \in \Lambda.$$

We have the following results about  $u(\gamma_t, x)$ .

**Lemma 3.2.**  $\forall t \leq s \leq T$ , we have  $u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) = Y^{\gamma_t, x}(s)$ .

**Proof.** For given  $\gamma_{t_1}$ ,  $t_1 < t$ , set  $X(t_1) = x$ . Consider the solution of FBSDE (2.1) and (2.2) on  $[t, T]$ :

$$\begin{aligned}
X^{\gamma_{t_1}, x}(s) &= X^{\gamma_{t_1}, x}(t) + \int_t^s b(W_r^{\gamma_{t_1}}, Y^{\gamma_{t_1}, x}(r), Z^{\gamma_{t_1}, x}(r)) dr + \int_t^s \sigma(W_r^{\gamma_{t_1}}, X^{\gamma_{t_1}, x}(r), Y^{\gamma_{t_1}, x}(r), Z^{\gamma_{t_1}, x}(r)) dW(r), \\
Y^{\gamma_{t_1}, x}(s) &= g(W_T^{\gamma_{t_1}}, X^{\gamma_{t_1}, x}(T)) - \int_s^T h(W_r^{\gamma_{t_1}}, X^{\gamma_{t_1}, x}(r), Y^{\gamma_{t_1}, x}(r), Z^{\gamma_{t_1}, x}(r)) dr - \int_s^T Z^{\gamma_{t_1}, x}(r) dW(r), \quad s \in [t, T].
\end{aligned}$$

For simplicity, set

$$\xi = X^{\gamma_{t_1}, x}(t).$$

Then we need to prove  $u(W_t^{\gamma_{t_1}}, \xi) = Y^{\gamma_{t_1}, x}(t)$ . Define

$$\begin{aligned}
W_t^{N, \gamma_{t_1}} &:= \sum_{i=1}^N I_{A_i} x_t^i, \\
\xi^N &:= \sum_{i=1}^N I_{A_i} a^i,
\end{aligned}$$

where  $\{A_i\}_{i=1}^N$  is a division of  $\mathcal{F}_t$ ,  $x_t^i \in A_i \cap \Lambda$ ,  $i = 1, 2, \dots, N$ . For any  $i$ ,  $(X^{x_t^i, a^i}(s), Y^{x_t^i, a^i}(s), Z^{x_t^i, a^i}(s))$  is the solution of the following FBSDE:

$$\begin{aligned} X^{x_t^i, a^i}(s) &= a^i + \int_t^s b(x_t^i, X^{x_t^i, a^i}(r), Y^{x_t^i, a^i}(r), Z^{x_t^i, a^i}(r))dr + \int_t^s \sigma(x_t^i, X^{x_t^i, a^i}(r), Y^{x_t^i, a^i}(r), Z^{x_t^i, a^i}(r))dW(r), \\ Y^{x_t^i, a^i}(s) &= g(W_T^{x_t^i}, X^{x_t^i, a^i}(T)) - \int_s^T h(x_t^i, X^{x_t^i, a^i}(r), Y^{x_t^i, a^i}(r), Z^{x_t^i, a^i}(r))dr - \int_s^T Z^{x_t^i, a^i}(r)dW(r), \quad s \in [t, T]. \end{aligned}$$

Multiplying by  $I_{A_i}$  and adding the corresponding terms, we obtain:

$$\begin{aligned} \sum_{i=1}^N I_{A_i} X^{x_t^i, a^i}(s) &= \sum_{i=1}^N I_{A_i} a^i + \int_t^s b(\sum_{i=1}^N I_{A_i} x_t^i, \sum_{i=1}^N I_{A_i} Y^{x_t^i, a^i}(r), \sum_{i=1}^N I_{A_i} Y^{x_t^i, a^i}(r), \sum_{i=1}^N I_{A_i} Z^{x_t^i, a^i}(r))dr \\ &\quad + \int_t^s \sigma(\sum_{i=1}^N I_{A_i} x_t^i, \sum_{i=1}^N I_{A_i} Y^{x_t^i, a^i}(r), \sum_{i=1}^N I_{A_i} Y^{x_t^i, a^i}(r), \sum_{i=1}^N I_{A_i} Z^{x_t^i, a^i}(r))dW(r), \\ \sum_{i=1}^N I_{A_i} Y^{x_t^i, a^i}(s) &= g(\sum_{i=1}^N I_{A_i} W_T^{x_t^i}, \sum_{i=1}^N I_{A_i} X^{x_t^i, a^i}(T)) - \int_s^T \sum_{i=1}^N I_{A_i} Z^{x_t^i, a^i}(r)dW(r), \\ &\quad - \int_s^T h(\sum_{i=1}^N I_{A_i} x_t^i, \sum_{i=1}^N I_{A_i} X^{x_t^i, a^i}(r), \sum_{i=1}^N I_{A_i} Y^{x_t^i, a^i}(r), \sum_{i=1}^N I_{A_i} Z^{x_t^i, a^i}(r))dr. \quad s \in [t, T]. \end{aligned}$$

So

$$X^{\gamma_t, x}(s) = \sum_{i=1}^N I_{A_i} X^{x_t^i, a^i}(s), \quad Y^{\gamma_t, x}(s) = \sum_{i=1}^N I_{A_i} Y^{x_t^i, a^i}(s), \quad Z^{\gamma_t, x}(s) = \sum_{i=1}^N I_{A_i} Z^{x_t^i, a^i}(s)$$

is the solution of the above FBSDE. By the definition of  $u$ , we get

$$Y^{W_t^{N, \gamma_{t_1}}, \xi^N}(t) = \sum_{i=1}^N I_{A_i} Y^{x_t^i, a^i}(t) = \sum_{i=1}^N I_{A_i} u(x_t^i, a^i) = u(W_t^{N, \gamma_{t_1}}, \xi^N).$$

For the general case, following the method in Peng and Wang [18] (Lemma 4.3), we choose a simple adapted process  $\{\gamma_t^i\}_{i=1}^\infty$  such that  $E \|\gamma_t^i - W_t^{\gamma_{t_1}}\|$  convergence to 0 as  $i \rightarrow \infty$ . Using the same procedure for  $\xi$ , we obtain

$$E \|Y^{\gamma_t^i, \xi^i}(t) - Y^{W_t^{\gamma_{t_1}}, \xi}(t)\|^2 \leq CE[\|\gamma_t^i - W_t^{\gamma_{t_1}}\| + \|\xi^i - \xi\|]$$

and

$$E \|u(\gamma_t^i, \xi^i) - u(W_t^{\gamma_{t_1}}, \xi)\|^2 \leq \bar{C}E[\|\gamma_t^i - W_t^{\gamma_{t_1}}\| + \|\xi^i - \xi\|].$$

This completes the proof.  $\square$

By Theorem 3.1, 3.2 and the definition of Dupire's vertical derivative, we have the following corollary.

**Corollary 3.1.**  $D_z u(\gamma_t, x)$  and  $D_{zz} u(\gamma_t, x)$  exist. Moreover,  $u(\gamma_t, x)$ ,  $D_z u(\gamma_t, x)$  and  $D_{zz} u(\gamma_t, x)$  are  $\Lambda$ -continuous.

**Proof.** By Theorem 3.2 we know that  $D_z u(\gamma_t, x)$  and  $D_{zz} u(\gamma_t, x)$  exist. In the following, we only prove  $u(\gamma_t, x)$  is  $\Lambda$ -continuous. The proof for the continuous property of  $D_z u(\gamma_t, x)$  and  $D_{zz} u(\gamma_t, x)$  is similar. Taking expectation on both sides of equation (2.2),

$$u(\gamma_t, x) = E g(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) - E \int_t^T h(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r))dr.$$

For  $\gamma_t, \bar{\gamma}_{\bar{t}} \in \Lambda$ ,  $\bar{t} \geq t$ , we have

$$\begin{aligned}
& |u(\gamma_t, x) - u(\bar{\gamma}_{\bar{t}}, x)| \\
& \leq E[|g(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) - g(W_T^{\bar{\gamma}_{\bar{t}}}, X^{\bar{\gamma}_{\bar{t}}, x}(T))|] + E[\int_t^{\bar{t}} |h(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r))| dr] \\
& \quad + E[\int_t^T |h(W_r^{\gamma_t}, X^{\gamma_t, x}(r), Y^{\gamma_t, x}(r), Z^{\gamma_t, x}(r)) - h(W_r^{\bar{\gamma}_{\bar{t}}}, X^{\bar{\gamma}_{\bar{t}}, x}(r), Y^{\bar{\gamma}_{\bar{t}}, x}(r), Z^{\bar{\gamma}_{\bar{t}}, x}(r))| dr] \\
& \leq E[C_1(1 + \|W_T^{\gamma_t}\|^k + \|W_T^{\bar{\gamma}_{\bar{t}}}\|^k) \|\gamma_t - \bar{\gamma}_{\bar{t}}\|] \\
& \quad + 3(\bar{t} - t)^{\frac{1}{2}} (\int_t^{\bar{t}} (|h(W_r^{\gamma_t}, 0, 0, 0)|^2 + |CX^{\gamma_t, x}(r)|^2 + |CZ^{\gamma_t, x}(r)|^2 + |CY^{\gamma_t, x}(r)|^2) dr)^{\frac{1}{2}} \\
& \quad + C \int_t^T (|X^{\gamma_t, x}(r) - X^{\bar{\gamma}_{\bar{t}}, x}(r)| + |Y^{\gamma_t, x}(r) - Y^{\bar{\gamma}_{\bar{t}}, x}(r)| + |Z^{\gamma_t, x}(r) - Z^{\bar{\gamma}_{\bar{t}}, x}(r)|) dr].
\end{aligned}$$

By Theorem 3.1, for some constant  $C_1$  depending only in  $C, k, x$  and  $T$ ,

$$|u(\gamma_t, x) - u(\bar{\gamma}_{\bar{t}}, x)| \leq C_1(1 + \|\gamma_t\|^k + \|\bar{\gamma}_{\bar{t}}\|^k)(\|\gamma_t - \bar{\gamma}_{\bar{t}}\| + |t - \bar{t}|^{\frac{1}{2}}).$$

This completes the proof.  $\square$

Using similar methods in this section, we can also prove that there exists the second order partial derivative of  $u$  in  $x$ . Finally, we have  $u \in \mathbb{C}^{0,2,2}(\Lambda \times \mathbb{R}^n)$ .

### 3.2 Path regularity of process $\mathbf{Z}$

In Pardoux and Peng [12], BSDE is only state-dependent, i.e.,  $h = h(t, \gamma(t), y, z)$  and  $g = g(\gamma(T))$ . Under appropriate assumptions,  $Y$  and  $Z$  are related in the following sense:

$$Z^{\gamma_t}(s) = \nabla_x u(s, \gamma_t(t) + W(s) - W(t)), \quad P - a.s.$$

Peng and Wang [18] extends this result to the path-dependent case. The corresponding BSDE is

$$Y^{\gamma_t}(s) = g(W_T^{\gamma_t}) - \int_s^T h(W_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) dr - \int_s^T Z^{\gamma_t}(r) dW(r), \quad s \in [t, T].$$

where  $W_T^{\gamma_t} = I_{s \leq t} \gamma_t(s) + I_{t < s \leq T}(\gamma_t(t) + W(s) - W(t))$ . Then under some assumptions, they obtained

$$Z^{\gamma_t, x}(s) = D_z u(W_s^{\gamma_t}), \quad P - a.s.$$

In this paper, we generalize it to the Non-Markovian fully coupled FBSDE case.

**Theorem 3.3.** *Under Assumptions 2.2, 3.2 and 3.3, for each  $\gamma_t \in \Lambda$ , the process  $(Z^{\gamma_t, x}(s))_{s \in [t, T]}$  has a continuous version with the form,*

$$Z^{\gamma_t, x}(s) = \sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) \nabla_x u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) + D_z u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)), \quad P - a.s.$$

To prove the above Theorem, we need the following lemma essentially from Pardoux and Peng [12].

**Lemma 3.3.** *Let  $\gamma_t$  and some  $\bar{t} \in [t, T]$  be given. Suppose that*

$$g(\gamma, z) = \varphi(\gamma(\bar{t}), \gamma(T) - \gamma(\bar{t}), z),$$

where  $\varphi$  is in  $C_p^3(\mathbb{R}^{2d} \times \mathbb{R}^m; \mathbb{R}^m)$ . For  $\phi = b, \sigma$  and  $h$ , suppose that

$$\phi(\gamma_t, x, y, z) = \bar{\phi}_1(s, \gamma_s(s), x, y, z) I_{[0, \bar{t})}(s) + \bar{\phi}_2(s, \gamma_s(\bar{t}), \gamma_s(s) - \gamma_s(\bar{t}), x, y, z) I_{[\bar{t}, T]}(s),$$

where  $\bar{\phi}_1, \bar{\phi}_2 \in C^{0,3}$ . Then for each  $s \in [t, T]$ ,

$$Z^{\gamma_t, x}(s) = \sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) \nabla_x u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) + D_z u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)), \quad P - a.s.$$

**Proof.** We only consider one dimensional case. For  $s \in [\bar{t}, T]$ , the FBSDE (2.1) and (2.2) can be rewritten as

$$\begin{aligned} X^{\gamma_s, x}(u) &= x + \int_s^u \bar{b}_2(r, \gamma_s(\bar{t}), W^{\gamma_s}(r) - \gamma_s(\bar{t}), X^{\gamma_s, x}(r), Y^{\gamma_s, x}(r), Z^{\gamma_s, x}(r))dr \\ &\quad + \int_s^u \bar{\sigma}_2(r, \gamma_s(\bar{t}), W^{\gamma_s}(r) - \gamma_s(\bar{t}), X^{\gamma_s, x}(r), Y^{\gamma_s, x}(r), Z^{\gamma_s, x}(r))dW(r), \quad u \in [s, T], \\ Y^{\gamma_t, x}(u) &= \varphi(\gamma_s(\bar{t}), W^{\gamma_s}(T) - \gamma_s(\bar{t}), X^{\gamma_s, x}(T)) \\ &\quad - \int_u^T \bar{h}_2(r, \gamma_s(\bar{t}), W^{\gamma_s}(r) - \gamma_s(\bar{t}), X^{\gamma_s, x}(r), Y^{\gamma_s, x}(r), Z^{\gamma_s, x}(r))dr - \int_u^T Z^{\gamma_s, x}(r)dW(r), \quad u \in [s, T]. \end{aligned}$$

For  $s \in [t, \bar{t}]$ ,

$$\begin{aligned} X^{\gamma_s, x}(u) &= x + \int_s^u \bar{b}_1(r, W^{\gamma_s}(r), X^{\gamma_s, x}(r), Y^{\gamma_s, x}(r), Z^{\gamma_s, x}(r))dr \\ &\quad + \int_s^u \bar{\sigma}_1(r, W^{\gamma_s}(r), X^{\gamma_s, x}(r), Y^{\gamma_s, x}(r), Z^{\gamma_s, x}(r))dW(r), \quad u \in [s, \bar{t}], \\ X^{\gamma_s, x}(u) &= X^{\gamma_s, x}(\bar{t}) + \int_{\bar{t}}^u \bar{b}_2(r, W^{\gamma_s}(\bar{t}), W^{\gamma_s}(r) - W^{\gamma_s}(\bar{t}), X^{\gamma_s, x}(r), Y^{\gamma_s, x}(r), Z^{\gamma_s, x}(r))dr \\ &\quad + \int_{\bar{t}}^u \bar{\sigma}_2(r, W^{\gamma_s}(\bar{t}), W^{\gamma_s}(r) - W^{\gamma_s}(\bar{t}), X^{\gamma_s, x}(r), Y^{\gamma_s, x}(r), Z^{\gamma_s, x}(r))dW(r), \quad u \in [\bar{t}, T], \end{aligned}$$

and

$$\begin{aligned} Y^{\gamma_s, x}(u) &= \varphi(W^{\gamma_s}(\bar{t}), W^{\gamma_s}(T) - W^{\gamma_s}(\bar{t}), X^{\gamma_s, x}(T)) \\ &\quad - \int_u^T \bar{h}_2(r, W^{\gamma_s}(\bar{t}), W^{\gamma_s}(r) - W^{\gamma_s}(\bar{t}), X^{\gamma_s, x}(r), Y^{\gamma_s, x}(r), Z^{\gamma_s, x}(r))dr - \int_u^T Z^{\gamma_s, x}(r)dW(r), \quad u \in [\bar{t}, T], \\ Y^{\gamma_s, x}(u) &= Y^{\gamma_s, x}(\bar{t}) - \int_u^{\bar{t}} \bar{h}_1(r, W^{\gamma_s}(\bar{t}), X^{\gamma_s, x}(r), Y^{\gamma_s, x}(r), Z^{\gamma_s, x}(r))dr - \int_u^{\bar{t}} Z^{\gamma_s, x}(r)dW(r), \quad u \in [s, \bar{t}]. \end{aligned}$$

Now consider the following quasilinear parabolic differential equations, which is defined on  $[\bar{t}, T] \times \mathbb{R}^2$  and parameterized by  $x \in \mathbb{R}$ ,

$$\begin{aligned} &\nabla_s u_2(s, x, y, z) + \mathcal{L}u_2(s, x, y, z) + \nabla_z \nabla_y u_2 \bar{\sigma}_2(s, x, y, z, u_2, v_2) + \frac{1}{2} \nabla_{yy} u_2 \\ &= \bar{h}_2(s, x, y, z, u_2, \nabla_y u_2(s, x, y) \bar{\sigma}_2(s, x, y, z, u_2, v_2)), \\ &v_2(s, x, y, z) = \nabla_z u_2(s, x, y, z) \bar{\sigma}_2(s, x, y, z, u_2, v_2) + \nabla_y u_2, \\ &u_2(T, x, y, z) = \varphi(x, y, z). \end{aligned}$$

where  $\mathcal{L} = \frac{1}{2} \bar{\sigma}_2^2 \nabla_{zz} + \bar{b}_2 \nabla_z$ . The other one is defined on  $[t, \bar{t}] \times \mathbb{R}$ ,

$$\begin{aligned} &\nabla_s u_1(s, x, z) + \mathcal{L}u_1(s, x, z) + \nabla_z \nabla_x u_1 \bar{\sigma}_1(s, x, z, u_1, v_1) + \frac{1}{2} \nabla_{xx} u_1 \\ &= \bar{h}_1(s, x, z, u_1, \nabla_z u_1 \bar{\sigma}_1(s, x, z, u_1, v_1)), \\ &v_1(s, x, z) = \nabla_z u_1 \bar{\sigma}_1(s, x, z, u_1, v_1) + \nabla_x u_1, \\ &u_1(\bar{t}, x, z) = u_2(\bar{t}, x, 0, z). \end{aligned}$$

where  $\mathcal{L} = \frac{1}{2} \bar{\sigma}_1^2 \nabla_{zz} + \bar{b}_1 \nabla_z$ . Following Theorem 3.1, 3.2 of Paroux-Peng [12], we have  $u_2 \in C^{1,2}([\bar{t}, T] \times \mathbb{R}^2; \mathbb{R})$ ,  $u_1 \in C^{1,2}([t, \bar{t}] \times \mathbb{R}; \mathbb{R})$  and

$$u(\gamma_s, z) = u_1(s, \gamma_s(s), z)I_{[t, \bar{t}]}(s) + u_2(s, \gamma_s(\bar{t}), \gamma_s(s) - \gamma_s(\bar{t}), z)I_{[\bar{t}, T]}(s).$$

By Itô formula and the uniqueness theorem of BSDE,

$$\begin{aligned} Y^{\gamma_t, x}(s) &= u_1(s, W^{\gamma_t}(s), X^{\gamma_t, x}(s)), \quad t \leq s < \bar{t}, \\ Y^{\gamma_t, x}(s) &= u_2(s, W^{\gamma_t}(\bar{t}), W^{\gamma_t}(s) - W^{\gamma_t}(\bar{t}), X^{\gamma_t, x}(s)), \quad \bar{t} \leq s \leq T, \\ Z^{\gamma_t, x}(s) &= \nabla_z u_1(s, W^{\gamma_t}(s)) \bar{\sigma}_1(s, W^{\gamma_t}(s), X^{\gamma_t, x}(s), u_1, Z^{\gamma_t, x}(s)) + \nabla_x u_1, \quad t \leq s < \bar{t}, \\ Z^{\gamma_t, x}(s) &= \nabla_z u_2(s, W^{\gamma_t}(\bar{t}), W^{\gamma_t}(s) - W^{\gamma_t}(\bar{t}), X^{\gamma_t, x}(s)) \\ &\quad \cdot \bar{\sigma}_2(s, W^{\gamma_t}(s), W^{\gamma_s}(s) - W^{\gamma_s}(\bar{t}), X^{\gamma_t, x}(s), u_2, Z^{\gamma_t, x}(s)) + \nabla_x u_2, \quad \bar{t} \leq s \leq T. \end{aligned}$$

Finally, for each  $s \in [t, T]$ ,

$$Z^{\gamma_t, x}(s) = \sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) \nabla_x u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) + D_z u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)), \quad P - a.s.$$

In particular, we have

$$Z^{\gamma_t, x}(t) = \sigma(\gamma_t, x, u(\gamma_t, x), Z^{\gamma_t, x}(t)) \nabla_x u(\gamma_t, x) + D_z u(\gamma_t, x). \quad \gamma_t \in \Lambda.$$

This completes the proof.  $\square$

Now we give the proof of Theorem 3.3.

**Proof.** For each fixed  $t \in [0, T]$  and positive integer  $n$ , we introduce a mapping  $\gamma^n(\bar{\gamma}_s, x) : \Lambda_s \mapsto \Lambda_s$ ,

$$\gamma^n(\bar{\gamma}_s)(r) = \bar{\gamma}_s(r) I_{[0, t]} + \sum_{k=0}^{n-1} \bar{\gamma}_s(t_{k+1}^n \wedge s) I_{[t_k^n \wedge s]}(r) + \bar{\gamma}_s(s) I_{\{s\}}(r), \quad s \in [0, T],$$

where  $t_k^n = t + \frac{k(T-t)}{n}$ ,  $k = 0, 1, \dots, n$ . Note that  $\Psi$  represents  $b$ ,  $\sigma$  or  $h$ . Define

$$g^n(\bar{\gamma}, x) := g(\gamma^n(\bar{\gamma})), \quad \Psi^n(\bar{\gamma}_s, x, y, z) := \Psi(\gamma^n(\bar{\gamma}_s), x, y, z).$$

For each  $n$ , there exists some functions  $\varphi_n$  defined on  $\Lambda_t \times \mathbb{R}^{n \times d}$  and  $\psi_n$  defined on  $[t, T] \times \Lambda_t \times \mathbb{R}^{n \times d} \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$  such that

$$\begin{aligned} g^n(\bar{\gamma}, x) &= \varphi_n(\bar{\gamma}_t, \bar{\gamma}(t_1^n) - \bar{\gamma}(t), \dots, \bar{\gamma}(t_n^n) - \bar{\gamma}(t_{n-1}^n), x), \\ \Psi^n(\bar{\gamma}_s, x, y, z) &= \psi_n(s, \bar{\gamma}_t, \bar{\gamma}_s(t_1^n \wedge s) - \bar{\gamma}_s(t), \dots, \bar{\gamma}_s(t_n^n \wedge s) - \bar{\gamma}_s(t_{n-1}^n \wedge s), x, y, z). \end{aligned}$$

Indeed, we can set

$$\begin{aligned} \bar{\varphi}_n(\bar{\gamma}_t, x_1, \dots, x_n, x) &:= g((\bar{\gamma}_t(s) I_{[0, t]}(s) + \sum_{k=1}^n x_k I_{[t_{k-1}^n, t_k^n]}(s) + x_n I_{\{T\}}(s))_{0 \leq s \leq T}, x), \\ \varphi_n(\bar{\gamma}_t, x_1, \dots, x_n, x) &:= \bar{\varphi}_n(\bar{\gamma}_t, \bar{\gamma}_t + x_1, \bar{\gamma}_t(t) + x_1 + x_2, \dots, \bar{\gamma}_t(t) + \sum_{i=1}^n x_i, x). \end{aligned}$$

Then by Assumptions 3.2 and 3.3, we obtain that  $\varphi_n(\bar{\gamma}_t, x_1, \dots, x_n, x)$  is a  $C_p^3$ -function of  $x_1, \dots, x_n$  for each fixed  $\bar{\gamma}_t$ . In particular, for each  $\bar{\gamma} \in \Lambda$ ,

$$\nabla_{x_i} \varphi_n(\bar{\gamma}_t, \bar{\gamma}(t_1^n) - \bar{\gamma}(t), \dots, \bar{\gamma}(t_n^n) - \bar{\gamma}(t_{n-1}^n), x) = g'_{\gamma_{t_{i-1}^n}}(\gamma^n(\bar{\gamma}), x).$$

For any  $\bar{t} \geq t$ ,  $\bar{\gamma}_{\bar{t}} \in \Lambda_{\bar{t}}$ , consider the following BSDEs:

$$Y^{n, \bar{\gamma}_{\bar{t}}}(s) = g^n(W_T^{\bar{\gamma}_{\bar{t}}}, X^{n, \bar{\gamma}_{\bar{t}}}(T)) - \int_s^T h^n(W_r^{\bar{\gamma}_{\bar{t}}}, X^{n, \bar{\gamma}_{\bar{t}}}(r), Y^{n, \bar{\gamma}_{\bar{t}}}(r), Z^{n, \bar{\gamma}_{\bar{t}}}(r)) dr - \int_s^T Y^{n, \bar{\gamma}_{\bar{t}}}(r) dW(r).$$

we denote

$$u^n(\bar{\gamma}_{\bar{t}}, x) := Y^{n, \bar{\gamma}_{\bar{t}}}(t), \quad \bar{\gamma}_{\bar{t}} \in \Lambda.$$

Following the argument as in Lemma 3.3, for each  $s \in [t, T]$ , we obtain

$$Z^{n, \bar{\gamma}_{\bar{t}}}(s) = \sigma^n(W_s^{\bar{\gamma}_{\bar{t}}}, X^{n, \bar{\gamma}_{\bar{t}}}(s), Y^{n, \bar{\gamma}_{\bar{t}}}(s), Z^{n, \bar{\gamma}_{\bar{t}}}(s)) \nabla_x u^n(W_s^{\bar{\gamma}_{\bar{t}}}, X^{n, \bar{\gamma}_{\bar{t}}}(s)) + D_z u^n(W_s^{\bar{\gamma}_{\bar{t}}}, X^{n, \bar{\gamma}_{\bar{t}}}(s)), \quad P - a.s.$$

Let  $C_0$  be a constant depends only on  $C, T$  and  $k, x$ , which allowed to change from line by line. Following the similar calculus as in Lemma 3.1 and Theorem 3.1, we get

$$\begin{aligned} & |u^n(\bar{\gamma}_{\bar{t}}, x) - u(\bar{\gamma}_{\bar{t}}, x)| \\ & \leq C_0 E[|g^n(W_T^{\bar{\gamma}_{\bar{t}}}, X^{\bar{\gamma}_{\bar{t}}, x}(T)) - g(W_T^{\bar{\gamma}_{\bar{t}}}, X^{\bar{\gamma}_{\bar{t}}, x}(T))| \\ & \quad + \int_t^T |h^n(W_r^{\bar{\gamma}_{\bar{t}}}, X^{n, \bar{\gamma}_{\bar{t}}, x}(r), Y^{n, \bar{\gamma}_{\bar{t}}, x}(r), Z^{n, \bar{\gamma}_{\bar{t}}, x}(r)) - h(W_r^{\bar{\gamma}_{\bar{t}}}, X^{\bar{\gamma}_{\bar{t}}, x}(r), Y^{\bar{\gamma}_{\bar{t}}, x}(r), Z^{\bar{\gamma}_{\bar{t}}, x}(r))|^2 dr]^{\frac{1}{2}} \\ & \leq C_0 (1 + \|\bar{\gamma}_{\bar{t}}\|^k) (\frac{1}{n^{\frac{1}{4}}} + \|\gamma^n(\bar{\gamma}_{\bar{t}}) - \bar{\gamma}_{\bar{t}}\|). \end{aligned}$$

and

$$\begin{aligned} |D_z u^n(\bar{\gamma}_{\bar{t}}, x) - D_z u(\bar{\gamma}_{\bar{t}}, x)| & \leq C_0 (1 + \|\bar{\gamma}_{\bar{t}}\|^k) (\frac{1}{n^{\frac{1}{4}}} + \|\gamma^n(\bar{\gamma}_{\bar{t}}) - \bar{\gamma}_{\bar{t}}\|), \\ |D_{zz} u^n(\bar{\gamma}_{\bar{t}}, x) - D_{zz} u(\bar{\gamma}_{\bar{t}}, x)| & \leq C_0 (1 + \|\bar{\gamma}_{\bar{t}}\|^k) (\frac{1}{n^{\frac{1}{4}}} + \|\gamma^n(\bar{\gamma}_{\bar{t}}) - \bar{\gamma}_{\bar{t}}\|). \end{aligned}$$

Since

$$\begin{aligned}
& \lim_n E[\sup_{s \in [t, T]} |D_z u^n(W_s^{\gamma_t}, X^{n, \gamma_t, x}(s)) - D_z u(W_s^{\gamma_t}, X^{\gamma_t, x}(s))|^2] \\
& \leq C_0 \lim_n E[\sup_{s \in [t, T]} |(1 + \|W_s^{\gamma_t}\|^k + \|X^{n, \gamma_t, x}(s)\|^k)(\frac{1}{n^{\frac{1}{4}}} + \|\gamma^n(W_s^{\gamma_t}) - W_s^{\gamma_t}\|)^2] \\
& \leq C_0 \lim_n (1 + \|\bar{\gamma}_t\|^k)(\frac{1}{n^{\frac{1}{4}}} + \|\gamma^n(\bar{\gamma}_t) - \bar{\gamma}_t\|) = 0
\end{aligned}$$

and  $\lim_n E[\int_t^T |Z^{\gamma_t, x}(u) - Z^{n, \gamma_t}(u)|^2 du] = 0$ , we have

$$Z^{\gamma_t, x}(s) = \sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) \nabla_x u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) + D_{zz} u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) \quad P - a.s.$$

This completes the proof.  $\square$

## 4 The related path-dependent partial differential equation

In this section, we relate FBSDE (2.1), (2.2) to the following path-dependent partial differential equation:

$$\begin{aligned}
& D_t u(\gamma_t, x) + \mathcal{L}u(\gamma_t, x) + tr[\nabla_x D_z u(\gamma_t, x) \sigma(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x))] + \frac{1}{2} tr[D_{zz} u(\gamma_t, x)] \\
& = h(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x)), \\
& v(\gamma_t, x) = \nabla_x u(\gamma_t, x) \sigma(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x)) + D_z u(\gamma_t, x), \\
& u(\gamma_T, x) = g(\gamma_T, x), \quad \gamma_T \in \Lambda^n,
\end{aligned} \tag{4.1}$$

where  $\nabla$  is the gradient operator and

$$\mathcal{L}u = \frac{1}{2} tr[(\sigma \sigma^T)(\gamma_t, x, u, v) \nabla_{xx} u] + \langle b(\gamma_t, x, u, v) \nabla_x u \rangle.$$

**Theorem 4.1.** *Suppose Assumptions 2.2, 3.2 and 3.3 hold. If  $u$  belongs to  $\mathbb{C}^{1,2,2}(\Lambda \times \mathbb{R})$  and  $(u, v)$  is the solution of equation (4.1) such that  $(u, v)$  is uniformly Lipschitz continuous and bounded by  $C(1 + |x| + \|\gamma_t\|)$ , then we have  $u(\gamma_t, x) = Y^{\gamma_t, x}(t)$ , for each  $(\gamma_t, x) \in \Lambda \times \mathbb{R}$ , where  $(X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s))_{t \leq s \leq T}$  is the unique solution of FBSDE (2.1), (2.2).*

**Proof.** By the assumptions of this theorem, we know that  $b(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x))$  and  $\sigma(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x))$  is uniformly Lipschitz continuous. Then the following SDE has a uniqueness solution.

$$\begin{aligned}
dX^{\gamma_t, x}(s) &= b(W_s^{\gamma_t}, X^{\gamma_t, x}(s), u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)), v(W_s^{\gamma_t}, X^{\gamma_t, x}(s))) ds \\
&\quad + \sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)), v(W_s^{\gamma_t}, X^{\gamma_t, x}(s))) dW(s), \\
X(t) &= x, \quad s \in [t, T].
\end{aligned}$$

Set

$$(Y(s), Z(s)) = (u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)), v(W_s^{\gamma_t}, X^{\gamma_t, x}(s))), \quad t \leq s \leq T.$$

We have

$$\begin{aligned}
dX^{\gamma_t, x}(s) &= b(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y(s), Z(s)) ds + \sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y(s), Z(s)) dW(s), \\
X(t) &= x, \quad s \in [t, T].
\end{aligned}$$

Note that  $u$  solves equation (4.1). Applying Functional Itô formula to  $Y(s) = u(W_s^{\gamma_t}, X^{\gamma_t, x}(s))$ , we get

$$\begin{aligned}
dY(s) &= h(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y(s), Z(s)) ds + Z(s) dW(s), \\
Y(T) &= g(W_T^{\gamma_t}, X^{\gamma_t, x}(T)) \quad s \in [t, T].
\end{aligned}$$

Thus, by the uniqueness and existence theorem of the FBSDE (Theorem 2.2), we have the result.  $\square$

Now we prove the converse to the about result.

**Theorem 4.2.** Under Assumptions 2.2, 3.2 and 3.3, the function  $u(\gamma_t, x) = Y^{\gamma_t, x}(t)$  is the unique  $\mathbb{C}^{1,2,2}(\Lambda \times \mathbb{R}^m)$ -solution of the path-dependent PDE (4.1).

**Proof.** We only study the one dimensional case. By Corollary 3.1,  $u \in \mathbb{C}^{0,2,2}(\Lambda)$ . Let  $\delta > 0$  be such that  $t + \delta \leq T$ . By Lemma 3.2, we get

$$u(X_{t+\delta}^{\gamma_t, x}) = Y^{\gamma_t, x}(t + \delta).$$

Hence

$$u(\gamma_{t,t+\delta}, x) - u(\gamma_t, x) = u(\gamma_{t,t+\delta}, x) - u(W_{t+\delta}^{\gamma_t}, X^{\gamma_t, x}(t + \delta)) + u(W_{t+\delta}^{\gamma_t}, X^{\gamma_t, x}(t + \delta)) - u(\gamma_t, x).$$

Similarly as the proof of Theorem 3.3, we obtain

$$\begin{aligned} & u(\gamma_{t,t+\delta}, x) - u(W_{t+\delta}^{\gamma_t}, X^{\gamma_t, x}(t + \delta)) \\ &= \lim_{n \rightarrow \infty} [u^n(\gamma_{t,t+\delta}, x) - u^n(W_{t+\delta}^{\gamma_t}, X^{\gamma_t, x}(t + \delta))] \\ &+ \int_t^{t+\delta} h(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) ds + \int_t^{t+\delta} Z^{\gamma_t, x}(s) dW(s). \end{aligned}$$

Following Lemma 3.1 and Theorem 3.2 of Pardoux and Peng [12] and Theorem 4.4 of Peng and Wang [18], we deduce that

$$\begin{aligned} & u^n(\gamma_{t,t+\delta}, x) - u^n(W_{t+\delta}^{\gamma_t}, X^{\gamma_t, x}(t + \delta)) \\ &= \int_t^{t+\delta} D_s u^n(\gamma_{t,s}, x) ds - \int_t^{t+\delta} D_s u^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) ds - \int_t^{t+\delta} D_z u^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) dW(s) \\ &- \frac{1}{2} \int_t^{t+\delta} D_{xx} u^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) ds - \frac{1}{2} \int_t^{t+\delta} \nabla_{xx} u^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) (\sigma^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)))^2 ds \\ &- \int_t^{t+\delta} D_z \nabla_x u^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) \sigma^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) ds \\ &- \int_t^{t+\delta} \nabla_x u^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) b^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) ds \\ &- \int_t^{t+\delta} \nabla_x u^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) \sigma^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) dW(s). \end{aligned}$$

Thus, by the dominated convergence theorem,

$$\begin{aligned} & u(\gamma_{t,t+\delta}, x) - u(\gamma_t, x) \\ &= - \int_t^{t+\delta} D_z u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) dW(s) - \frac{1}{2} \int_t^{t+\delta} D_{xx} u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) ds \\ &- \frac{1}{2} \int_t^{t+\delta} \nabla_{xx} u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) (\sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)))^2 ds \\ &- \int_t^{t+\delta} D_z \nabla_x u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) \sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) ds \\ &- \int_t^{t+\delta} \nabla_x u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) \sigma(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) dW(s) \\ &- \int_t^{t+\delta} \nabla_x u(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) b(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) ds \\ &+ \int_t^{t+\delta} h(W_s^{\gamma_t}, X^{\gamma_t, x}(s), Y^{\gamma_t, x}(s), Z^{\gamma_t, x}(s)) ds + \int_t^{t+\delta} Z^{\gamma_t, x}(s) dW(s) + \lim_{n \rightarrow \infty} C^n, \end{aligned} \tag{4.2}$$

where

$$C^n = \int_t^{t+\delta} D_s u^n(\gamma_{t,s}, x) ds - \int_t^{t+\delta} D_s u^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s)) ds.$$

Since  $u^n(\gamma_t, x) \in \mathbb{C}_{l, lip}^{0,2,2}(\Lambda \times \mathbb{R})$ , by Lemma 3.3 we get

$$|D_s u^n(\gamma_{t,s}, x) - D_s u^n(W_s^{\gamma_t}, X^{\gamma_t, x}(s))| \leq c(\|\gamma_{t,s} - W_s^{\gamma_t}\| + |X^{\gamma_t, x}(s) - x|)$$

for some constant  $c$  only depending on  $C, T, \gamma_t$  and  $k$ . Hence

$$|C^n| \leq c\delta \left( \sup_{s \in [t, t+\delta]} |W^{\gamma_t}(s) - \gamma_t(t)| + |X^{\gamma_t, x}(s) - x| \right).$$

Taking expectation on both sides of (4.2), we have

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{u(\gamma_{t,t+\delta}, x) - u(\gamma_t, x)}{\delta} \\ &= -\mathcal{L}u(\gamma_t, x) - (\nabla_x D_z u(\gamma_t, x) \sigma(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x)) + \frac{1}{2} D_{xx} u(\gamma_t, x)) + h(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x)), \end{aligned}$$

where

$$v(\gamma_t, x) = \nabla_x u(\gamma_t, x) \sigma(\gamma_t, x, u(\gamma_t, x), v(\gamma_t, x)) + D_z u(\gamma_t, x).$$

Thus,  $u(\gamma_t, x) \in \mathbb{C}^{1,2,2}(\Lambda \times \mathbb{R})$  satisfies the equation (4.1).  $\square$



## References

- [1] J. M. Bismut, Conjugate convex functions in optimal stochastic control, *Journal of Mathenmatical Analysis and Applications*, 44, 384-404.
- [2] R. Cont and D. A. Fourni, Functional Itô calculus and stochastic intergral representation of martingales, in [arxiv.org/abs/1002.2446](https://arxiv.org/abs/1002.2446), 2010.
- [3] R. Cont and D. A. Fourni, A functional extension of the Ito formula, *C. R. Math. Acad. Sci. Paris* 348 (2010), no. 1-2, 57-61.
- [4] R. Cont and D. A. Fourni, Change of variable formulas for non-anticipative functionals on path space, *J. Funct. Anal.* 259 (2010), no. 4, 1043-1072.
- [5] A. Cosso, Viscosity solutions of path-dependent PDEs and non-Markovian forward-backward stochastic systems, Preprint, 2012.
- [6] B. Dupire, Functional Itô calculus, Portfolio Research Paper, Bloomberg, 2009.
- [7] I. Ekren, C. Keller, N. Touzi and J. Zhang, On Viscosity Solutions of Path Dependent PDEs, [arXiv:1109.5971](https://arxiv.org/abs/1109.5971), 2011.
- [8] Y. Hu and S. Peng, Solution of a forward-backward stochastic differential equations, *Probability Theory and Related Fields*, 103, 273-283, 1995.
- [9] J. Ma, P. Protter and J. Yong, Solving forward-backward stochastic differential equations explicitly—a four step scheme, *Probablity Theory and Related Fields*, 98, 339-359, 1994.
- [10] J. Ma and J. Yong, Forward-backward Stochastic Differential Equations and Their Applications, *Lecture Notes in Math*, 1702, Springer, 1999.
- [11] E. Pardoux and S. Tang. Forward-backward stochastic differential equations and quasilinear parabolic PDEs, *Probablity Theory and Related Fields*, 114, no.2, 123-150, 1999.
- [12] E. Pardoux and S. Peng, Backward stochastic equations and quasilinear parabolic partial differential equation, In: B. L. Rozuvskii and R. B. Soeers (eds), *Stochastic partial diferential equation and their applications*, (Lect. Notes Control Inf. Sci., vol. 176,200-217) Berlin Heidelberg New York: Springer.
- [13] E. Pardoux and S. Peng, Adapted Solutions of Backward Stochastic Equations, *Systerm and Control Letters*, 14, 55-61.
- [14] S. Peng, Probabilistic Interpretation for Systems of Quasilinear Parabolic Partial Differential Equation, *Stochastic*, 37, 61-74.
- [15] S. Peng and Zhen Wu, Fully coupled forward-backward stochastic differential equations and applications to optimal control, *SIAM J. CONTROL OPTIM*, 37(3), 825-843, 1999.
- [16] S. Peng, Backward Stochastic Differential Equation, Nonlinear Expectation and Their Application, *Proceedings of the International congress of Mathematicians Hyderabad, India*
- [17] S. Peng, Note on Viscosity Solution of Path-Dependent PDE and G-Martingales, in [arxiv.org/abs/1106.1144v1](https://arxiv.org/abs/1106.1144v1).
- [18] S. Peng, Falei Wang, BSDE, Path-dependent PDE and Nonlinear Feynman-Kac Formula, in [arxiv.org/abs/1108.4317](https://arxiv.org/abs/1108.4317).